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Direct asymptotic analysis of the second Painlevé equation: three different limits

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Abstract. We present three different asymptotic studies of the second Painlevé equation $y'' = 2y^3 + xy + \alpha$ involving $|x| \to \infty$, $|\alpha| \to \infty$ or unbounded initial data. We show how the direct method, which is in the spirit of Boutroux, can be naturally applied to each of the three cases.

1. Introduction

The six Painlevé equations are well known nonlinear second-order ordinary differential equations (ODEs) in the complex plane. These equations exhibit the Painlevé property which states that the only possible movable singularities in their solutions are poles. They were identified by Painlevé [23], Gambier [10] and Fuchs [9] as the only second-order ODEs (with degree one) with the Painlevé property whose generic solutions are new transcendental functions.

The Painlevé property is strongly linked to complete integrability (in the sense of solvability through an associated single-valued linear system). Ablowitz *et al* [1, 2] have shown extensive evidence that ODE reductions of completely integrable partial differential equations necessarily have the Painlevé property.

The Painlevé transcendents play a distinguished role as nonlinear special functions. In particular, they can be written in terms of entire functions [22], they are isomonodromy conditions for associated linear ODEs [9] and their connection problem can be solved directly [17].

The historical development of the classical special functions [21] is interwoven with their asymptotic analysis. Indeed, asymptotics provides a deeper understanding of such functions. This is also true of the Painlevé transcendents. In particular, asymptotic analysis provides the only known explicit description of these transcendents in terms of known classical functions.

The asymptotic description of the Painlevé equations in the limit as their independent variable tends to a fixed singularity, such as infinity, has been well studied [5, 6, 3, 24, 16, 7, 17, 14]. An important problem here is to relate the solution near a fixed singularity to its behaviour near another (or the same) such singularity. The case when a *parameter* of the equations approaches infinity has only recently been studied [19, 15]. The

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double asymptotic limit when the independent variable and a parameter approaches infinity is given in [20]. The results in [20, 19] are restricted to formal special solutions.

Of the various different methodologies used to determine the asymptotic behaviour of the Painlevé transcendents, the most natural appears to be that of Boutroux [5, 6] and of Joshi and Kruskal [17, 18, 15]. We use their direct method here to study the second Painlevé equation, PII, given by

$$y'' = 2y^3 + xy + \alpha. \tag{1}$$

This equation is encountered in several physical applications such as a spherical electric probe in continuum plasma [8], Görtler vortices in boundary layers [12, 13, 4] and nonlinear optics [11]. In section 2, we describe Boutroux's result for unbounded $|x|$ and announce Joshi's result for unbounded $|\alpha|$. In section 3, we give details of a new result in the case of unbounded initial data. Our result also applies to those physical applications where large but finite data may be given. Another reason for studying such a limit lies in the fact that the solution space of a nonlinear ODE cannot be completely described without studying the limits of known behaviours. Unbounded initial data provides one such limit. In all three cases, the generic solution is given by elliptic functions to leading order.

2. Direct asymptotic method for PII

The direct asymptotic method involves the following steps:

(1) transform equation according to dominant balance;
(2) integrate dominant terms;
(3) establish generic conditions;
(4) estimate error of leading-order approximate solution.

Let the new transformed variables be (z, u) after step 1. The Painlevé equations are second-order ODEs, so one has a unique solution $u(z)$ for any given bounded initial data (not corresponding to singular values of the equation):

$$(z, u, u_z) = (z_0, \eta, \eta'). \tag{2}$$

The leading-order approximation of PII can then be shown to be given in terms of elliptic functions defined implicitly by

$$\int_\Gamma \frac{\mathrm{d}v}{\sqrt{P(v)}} = z - z_0 \tag{3}$$

where P is some quartic polynomial with roots d_i, $i = 1, 2, 3, 4$, and Γ is any path in the u-plane from η to u (avoiding d_i). These elliptic functions possess two periods given by

$$\omega_j = \oint_{C_j} \frac{\mathrm{d}v}{\sqrt{P(v)}} \tag{4}$$

where C_j are linearly independent closed contours enclosing two roots of $P(u)$. Figure 1 illustrates the described quantities. See [15] for further properties of elliptic integrals.

The generic conditions usually involve upper bounds on:

- length of Γ;
- $|u(z)|$,

and lower bounds on:

- distance between Γ and the roots of P;
- distance between roots of P.

We call such conditions generic since in the appropriate asymptotic limit, these lower or upper bounds can be made, respectively, as small or as large as one requires. For the

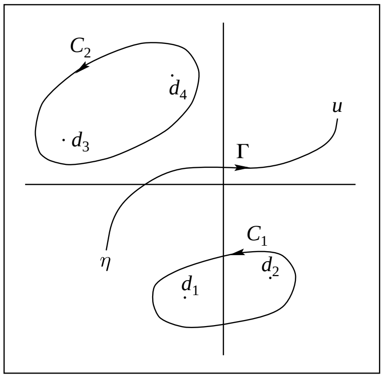

Figure 1. Integration curves in u-plane for PII.

sake of clarity, we do not explicitly state what the generic assumptions are in the following illustrative examples. For this and the proof of the results, the reader is referred to the relevant references. However, complete details for our new result is provided in section 3.

Example 2.1. We first illustrate the direct method by describing Boutroux's asymptotic estimate [5, 6] for PII in the limit $|x| \to \infty$. In step 1, the dominant balance argument [17] for this case leads to a change of variables

$$y(x) = \sqrt{x}u(z) \qquad z = \tfrac{2}{3}x^{3/2}$$

which transforms PII to

$$u_{zz} = 2u^3 + u + \frac{1}{z}\left(\frac{2\alpha}{3} - u_z + \frac{u}{9z}\right).$$

The dominant terms are u_{zz} and $2u^3 + u$. In step 2 we integrate these dominant terms to obtain

$$u_z^2 = P(u) + S$$

where

$$P(u) = u^4 + u^2 + 2E \qquad S = 2\int_{z_0}^{z} \frac{u_z}{z}\left(\frac{2\alpha}{3} - u_z + \frac{u}{9z}\right) \mathrm{d}z$$

and E depends on the initial data (2), kept fixed in the analysis. As z gets large, S becomes small and the leading-order solution in step 3 is a (Jacobian) elliptic function given by (3) with P given above. Under certain generic conditions [5, 6] which involve $|z| > 1/\epsilon$, $\epsilon > 0$, the required error estimate between this leading-order solution and the true solution is given by the following theorem.

Theorem 2.2. Under generic conditions, $\exists\ k$, $\epsilon_0 > 0$ s.t. for $0 < \epsilon < \epsilon_0$, we have

$$\left|\int_\eta^u \frac{\mathrm{d}v}{\sqrt{P(v)}} - (z - z_0)\right| < k\sqrt{\epsilon}.$$

Moreover, if z_0 and z_1 are two successive points where $u = \eta$, then for $j = 1$ or 2,

$$|(z_0 - z_1) - \omega_j| < k\sqrt{\epsilon}$$

where ω_j are the two periods given by (4).

Example 2.3. In this example, we announce Joshi's [15] asymptotic estimate for PII in the limit $|\alpha| \to \infty$. In step 1, the dominant balance argument for this case leads to a change of variables

$$y(x) = \epsilon^{-1/3} u(z) \qquad z = \epsilon^{-1/3} x \qquad \epsilon = \alpha^{-1}$$

which transforms PII to

$$u_{zz} = 2u^3 + 1 + \epsilon z u.$$

The dominant terms are u_{zz} and $2u^3 + 1$. In step 2 we integrate these dominant terms to obtain

$$u_z^2 = P(u) + \epsilon S$$

where

$$P(u) = u^4 + 2u + 2E \qquad S = 2\int_{z_0}^{z} z u u_z \, \mathrm{d}z$$

and E depends on the initial data, kept fixed in the analysis. For this P, the leading-order solution in step 3 is a (Jacobian) elliptic function given by (3). Under certain generic conditions [15], the required error estimate between this leading-order solution and the true solution is as follows.

Theorem 2.4. Under generic conditions, $\exists\, \epsilon_0 > 0$ s.t. for $0 < \epsilon < \epsilon_0$, we have

$$\left| \int_{\eta}^{u} \frac{\mathrm{d}v}{\sqrt{P(v)}} - (z - z_0) \right| < \sqrt{2}\epsilon^{1/2} |\log \epsilon|.$$

Moreover, if z_0 and z_1 are two successive points where $u = \eta$, then for $j = 1$ or 2,

$$|(z_0 - z_1) - \omega_j| < \sqrt{2}\epsilon^{1/2} |\log \epsilon|$$

where ω_j are the two periods given by (4).

3. Unbounded initial data for PII

Consider the second Painlevé equation, PII, given by (1), when the parameter α is fixed and the given initial conditions are $(x, y, y') = (x_0, y_0, y_0')$. Multiply (1) by y' and integrate to obtain

$$y'^2 = y^4 + 2\alpha y + 2\int_{x_0}^{x} x y y' \, \mathrm{d}x + E \tag{5}$$

where

$$E = (y_0')^2 - y_0^4 - 2\alpha y_0.$$

By large initial data, we mean large $|E|$. Hence rescale according to

$$y = \epsilon^{-1/3} u(z) \qquad z = \epsilon^{-1/3} x \qquad \epsilon = E^{-3/4}.$$

Then (5) becomes

$$u_z^2 = P(u) + \epsilon S \tag{6}$$

where

$$P(u) = u^4 + 1 \qquad S = 2\alpha u + 2\int_{z_0}^{z} z u u_z \, \mathrm{d}z. \tag{7}$$

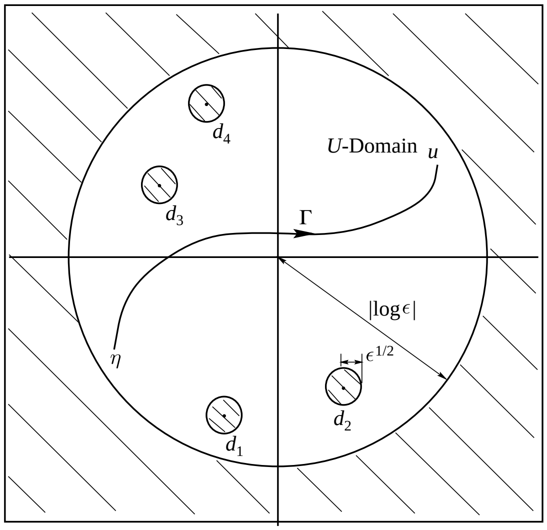

Figure 2. Integration curve in the U-domain for PII.

The transformed initial conditions are

$$(z, u, u_z) = (z_0, \eta, \eta') = (\epsilon^{-1/3}x_0, \epsilon^{1/3}y_0, \epsilon^{2/3}y_0'). \tag{8}$$

Our aim is to show that the solutions of (5) in the case of small ϵ can be approximated by the (Jacobi) elliptic functions given implicitly by (3) with two periods (4), where P is given by (7). The roots of P are

$$d_k = (-1)^{1/4} \qquad k = 1, 2, 3, 4.$$

In particular, note that the distance between any two roots satisfies

$$|d_i - d_j| \geqslant \sqrt{2} \qquad i \neq j. \tag{9}$$

Definition 3.1. The following are called *generic conditions*.

(1) $0 < \epsilon < \frac{1}{16}$, $|\alpha| < |\log \epsilon|$.

(2) (U-domain) $|u| < |\log \epsilon|$ and $\epsilon^{1/2} < |u - d_k|$, $k = 1, 2, 3, 4$.

(3) (Z-domain) $|z| < |\log \epsilon|$.

(4) Initial conditions (8) such that $(z_0, \eta) \in Z \times U$.

(5) Γ is any path in U, from η to w, such that its length $|\Gamma| < |\log \epsilon|$.

Figure 2 depicts the U-domain.

Definition 3.2. Suppose ϵ and the initial conditions (8) are given which satisfy the generic conditions. Then the unique solution of (6) satisfying these initial conditions is called a *generic solution* of PII.

Theorem 3.3. Under generic conditions, $\exists\ 0 < \epsilon_0 < \frac{1}{16}$ s.t. for $0 < \epsilon < \epsilon_0$, we have

$$\left| \int_\Gamma \frac{\mathrm{d}v}{\sqrt{P(v)}} - (z - z_0) \right| < 3\sqrt{2}\epsilon^{1/4} |\log \epsilon|^4. \tag{10}$$

Moreover, the distance between any two successive points z_0 and z_1 where $u = \eta$ satisfies

$$|(z_0 - z_1) - \omega_j| < 3\sqrt{2}\epsilon^{1/4} |\log \epsilon|^4 \qquad j = 1, 2. \tag{11}$$

Note that as $\epsilon \to 0$, the upper bounds in this theorem also tend to zero. In particular, for all $0 < \epsilon < 10^{-80}$, these upper bounds are less than $\frac{1}{2}\epsilon^{1/8}$. We emphasize that, within the constraints of the generic conditions, the above estimate is independent of the choice of path Γ.

Proof.

$$\begin{aligned} I &= \int_\eta^u \frac{\mathrm{d}v}{\sqrt{P(v)}} - (z - z_0) \\ &= \int_\eta^u \left(\frac{v_z}{\sqrt{P(v)}} - 1 \right) \frac{\mathrm{d}v}{v_z} \\ &= \int_\eta^u \left(\sqrt{1 + \frac{\epsilon S}{P}} - 1 \right) \frac{\mathrm{d}v}{v_z}. \end{aligned} \tag{12}$$

Here we have used the square root of (6), fixing the branch of the square root function to have values in the right half complex plane. That is,

$$\mathrm{Re}\sqrt{1 + \frac{\epsilon S}{P}} \geqslant 0. \tag{13}$$

What remains is to determine estimates for the various expressions in the integrand of (12). We do this through the following lemmas. □

Lemma 3.4. Under the generic assumptions

$$|P(u)| > \epsilon^{1/2}. \tag{14}$$

Proof. In the U-domain, $\inf |P(u)|$ occurs where u is closest to one of the roots of $P(u)$, i.e. $|u - d_k| = \epsilon^{1/2}$ for some $k = 1, 2, 3, 4$. Also, since the minimum distance between any two roots of $|P(u)|$ is given by (9), we arrive at

$$\begin{aligned} |P(u)| &> \epsilon^{1/2} |\sqrt{2} - \epsilon^{1/2}|^3 \\ &> \epsilon^{1/2}. \end{aligned}$$

We have also used the fact that $\epsilon < \frac{1}{16}$. □

Lemma 3.5. Under the generic assumptions

$$\left| \frac{\epsilon S}{P} \right| < 3\epsilon^{1/2} |\log \epsilon|^3. \tag{15}$$

Proof. The generic assumptions and (7) lead to

$$\begin{aligned} |S| &< |2\alpha u| + 2 \left| \int_{z_0}^z z u u_z \, \mathrm{d}z \right| \\ &< 2|\log \epsilon|^2 + 2 \left| \int_\eta^u z v \, \mathrm{d}v \right| \\ &< 3|\log \epsilon|^3. \end{aligned}$$

This together with (14) lead to the desired result (15). □

Lemma 3.6. Under the generic assumptions

$$\left| \sqrt{1 + \frac{\epsilon S}{P}} - 1 \right| < 3\epsilon^{1/2} |\log \epsilon|^3. \tag{16}$$

Proof. Let

$$Q = \sqrt{1 + \frac{\epsilon S}{P}} + 1 \tag{17}$$

then the left-hand side of (16) is just

$$|Q - 2| = \left|\frac{\epsilon S}{PQ}\right|.$$

The desired result is obtained from the upper bound for $|\epsilon S/P|$, given by (15), and the lower bound $|Q| > 1$ which follows from the previously chosen branch (13) of the square root in (17). □

Lemma 3.7. Under the generic assumptions, there exists $0 < \epsilon_0 < \frac{1}{16}$ such that for all $0 < \epsilon < \epsilon_0$,

$$|u_z| > \frac{\epsilon^{1/4}}{\sqrt{2}}.$$

Proof. From (6), we have

$$|u_z^2| > |P|(1 - |\epsilon S/P|).$$

Using (14) and (15), we have

$$|u_z^2| > \epsilon^{1/2}(1 - 3\epsilon^{1/2}|\log\epsilon|^3).$$

It is easy to show that there exists $0 < \epsilon_0 < \frac{1}{16}$ such that $3\epsilon^{1/2}|\log\epsilon|^3 < \frac{1}{2}$ and the lemma is proved. □

Finally, we can complete the proof of the theorem. By lemmas 3.6 and 3.7 and (12), we have

$$\begin{aligned} |I| &< \frac{3\epsilon^{1/2}|\log\epsilon|^3}{\epsilon^{1/4}/\sqrt{2}}|\log\epsilon| \\ &< 3\sqrt{2}\epsilon^{1/4}|\log\epsilon|^4. \end{aligned}$$

This proves (10). Estimate (11), for the distance between two successive points z_0 and z_1 where $u = \eta$, is given by a similar argument, where Γ is chosen to be a closed curve enclosing two roots d_k of $P(u)$, and by using (4). □

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